# Twist free energy in a spin glass

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**Abstract.** The field theory of a short range spin glass with Gaussian random interactions, is considered near the upper critical dimension six. In the glassy phase, replica symmetry breaking is accompanied with massless Goldstone modes, generated by the breaking of reparametrization invariance of a Parisi type solution. Twisted boundary conditions are thus imposed at two opposite ends of the system in order to study the size dependence of the twist free energy. A loop-expansion is performed to first order around a twisted background. It is found, as expected but it is non trivial, that the theory does renormalize around such backgrounds, as well as for the bulk. However two main differences appear, in comparison with simple ferromagnetic transitions: (i) the loop expansion yields a (negative) anomaly in the size dependence of the free energy, thereby lifting the lower critical dimension to a value greater than two (ii) the free energy is lowered by twisting the boundary conditions. This situation is common in spin glasses, reflecting the non-positivity of mode multiplicity in replica symmetry breaking, but its physical meaning is still unclear.

**PACS.** 64.70.Pf Glass transitions – 64.60.Cn Order-disorder transformations; statistical mechanics of model systems

## 1 Introduction

Spontaneously broken symmetries are characterized by the existence of several possible pure states. If one imposes "twisted" boundary conditions, *i.e.* different pure states at two ends of the system, the free energy per unit volume will be slightly greater than the free energy corresponding to one single pure state over the whole system.

For a simple discrete symmetry, such as the $Z_2$-symmetry of Ising-like systems, one may consider an (hyper)-cubic system with up spins in the $z = 0$ plane, down spins in the $z = L$ plane and for instance periodic boundary conditions in the transverse directions $x_1, x_2, \cdots x_{d-1}$. This will generate an interface in the system centered around some plane $z = z_0$ and a cost in free energy

$$\Delta F = F_{\uparrow,\downarrow} - F_{\uparrow,\uparrow} = \sigma L^{d-1} \tag{1}$$

in which $\sigma(T)$ is the interfacial tension. As is well-known the power $(d-1)$ of $L$ in (1) implies that the lower critical dimension of systems with a discrete symmetry is equal to one, *i.e.* there is no ordered phase unless $d$ is greater than one. At leading order the classical (mean field) configuration for the order parameter, given the boundary conditions, is a kink of hyperbolic tangent shape, interpolating

between up and down spins. The fluctuations are given at one-loop order by the Fredholm determinant of a one-dimensional Schrödinger operator in a $1/\cosh^2(z - z_0)$ potential [2] which, as is well-known, is solvable analytically. Every term in the loop expansion for the free energy about mean field theory, is then proportional to $L^{d-1}$, and the successive contributions build up the correct exponent and amplitude for the interfacial tension $\sigma$.

For continuum spontaneously broken symmetries, the situation about the upper critical dimension is technically different. For an $N$-vector model one considers for definiteness an order parameter, which is uniform along the vector $(1, 0, \cdots, 0)$ in the $z = 0$ plane, and uniform but rotated by an angle $\theta_0$ in the plane $z = L$, *i.e.* lying along the vector $(\cos\theta_0, \sin\theta_0, 0, \cdots, 0)$. There again one expects a cost in free energy

$$\Delta F = \sigma(T, \theta_0)L^{d-2} \tag{2}$$

in agreement with a lower critical dimension equal to two, and with a "twist" energy $\sigma(T, \theta_0)$ (or spin stiffness constant) vanishing as $\theta_0^2$ for small $\theta_0$, (the ratio $\sigma/\theta_0^2$ is the helicity modulus [3]). If it is quite elementary to verify these statements within mean field theory, not difficult also to check them in the vicinity of the lower critical dimension $d_l = 2$ through the non-linear sigma model [4,5]. Near the upper critical dimension $d_u = 4$, things are not as simple. The mean field solution is not elementary and one may fear that the loop expansion might be difficult

to handle. However it turns out [6] that for $L$ large, the analysis of fluctuations is simply perturbative and finally explicit. It follows from this analysis that the massless Goldstone modes give, as expected, an $L^{d-2}$ behaviour in the twist free energy to all orders in the loop expansion.

For a spin glass the nature of the broken symmetry in the low temperature phase is more difficult to visualize. However within the replica approach, and Parisi's ansatz for the mean field solution [7], there are indeed "replicon" massless Goldstone modes [8] (plus "anomalous" massless modes). The broken symmetry at the origin of those modes may be related to a *reparametrization invariance* of the action. More specifically the mean field solution depends, in the continuum limit of Parisi's scheme of replica symmetry breaking for the Edwards-Anderson model, of two functions $p(t)$ and $Q(t)$ in which $t$ is the continuum labelling of the steps of breaking, $Q(t)$ the Parisi order parameter and $p(t)$, the continuum limit of the size of the successive boxes in which the $n$ replicas are divided. The free energy is not a separate function of $Q(t)$ and $p(t)$ but depends only of $Q$ as a function of $p$, leading for instance to the simple "gauge choice" $p(t) = t$ of Parisi. The existence of massless modes may be related to this arbitrariness [12]. However one does not see any physical "external field" which could be used to tune a given specific gauge choice. In particular there is no *a priori* connection between a twist of the boundary conditions on the spins themselves, with that of the twist on the gauge choice. Therefore the size dependence studied in this article, is not necessarily related to the usual stiffness exponent, measured in numerical simulations for instance. The situation is somewhat reminiscent of cases such as superfluid Helium, in which there is no physical conjugate variable to the order parameter which one could use to fix its phase. However if one takes two samples, they have no reason to carry the same phase, and this phase difference manifest itself in Josephson's junctions for instance.

In this note we report the result of an analysis, in which one imposes again two different schemes at two ends of the system. In the $z = 0$ plane we have chosen the simple Parisi gauge

$$p(t, z = 0) = t \tag{3}$$

whereas in the $z = L$ plane we have imposed

$$p(t, z = L) = t + h(t) \tag{4}$$

in which we assume that $h(t)$ is some given infinitesimal function, vanishing with $t$, with support $0 \leq t \leq \tilde{x}$. All calculations have been performed to lowest order in $h(t)$. The mean field solution, to lowest order in $h(t)$, provides a linear interpolation between the two end planes, and a free energy which is proportional to $L^{d-2}$ as for the N-vector model. At one-loop order, in dimension $d = 6 - \epsilon$ one finds after a long calculation, whose details will be reported elsewhere, a free energy for the twist (3, 4) which is proportional to $L^{d-2} \log L$. Those logarithms, which are caused here by the absence of a mass gap to the Goldstone modes, change drastically the situation compared with ordered states. They may be exponentiated in the standard

renormalization group way and yield, to first order in $\epsilon$ a twist free energy which is proportional to

$$\Delta F \simeq -\tau^{2+\varepsilon} L^{d-2-\varepsilon/3} \int_0^{\tilde{x}} \mathrm{d}t\ h^2(t), \tag{5}$$

in which $\tau$ measures the temperature below the glassy transition. This may be identified, up to one-loop, as an approximation to

$$\Delta F \simeq -\left(\frac{L}{\xi}\right)^{d-2+\eta}. \tag{6}$$

The occurence of fluctuations which modify the size dependence of the mean field result, implies a renormalization group procedure to interpret them. Therefore, contrary to ordered transitions, we are limited to a region below $T_c$ in which the correlation length $\xi$ is large.

Since $\eta$ is negative, this shows that the lower critical dimension $d_c$ is larger than two. If the previous identification of the $-\epsilon/3$ being the one-loop approximation to the $\eta$ exponent is correct, then the lower critical dimension is given by

$$d_c = 2 - \eta(d_c). \tag{7}$$

However this is simply a bold extrapolation of our one-loop result. Let us stress again that the stiffness exponent computed here is relative to twisting the gauge choice $p(z)$. Unfortunately there is no physical field conjugate to $p(z)$ which would allow a comparison with numerical estimates. The sign of the result is a puzzle on which we have a few comments at the end.

## 2 Mean field theory

The action for the Edwards-Anderson spin glass is written in terms of an $n \times n$ matrix $Q_{ab}$, in which $a$ and $b$ are replica indices and

$$S = \int \mathrm{d}^d x \left\{ \sum_{ab} \left( \frac{1}{4} (\nabla Q_{ab}(x))^2 + \frac{\tau}{2} Q_{ab}^2 + \frac{u}{12} Q_{ab}^4 \right) \right.$$
$$\left. + \frac{w}{6} \sum_{abc} Q_{ab} Q_{bc} Q_{ca} \right\}. \tag{8}$$

In a Parisi replica infinite symmetry breaking scheme, one divides the $n = p_0$ replicas into $p_0/p_1$ boxes of size $p_1$; each box of size $p_1$ is divided into $p_1/p_2$ boxes of size $p_2$, and so on, ad infinitum. The matrix elements $Q_{ab}$ follow those steps and are characterized by a correlative infinite sequence $Q_0, Q_1, \cdots$

In the continuum limit, we are thus led to an action which depends on two spatially varying functions $p(t, z)$ and $Q(t, z)$, in which $0 < z < L$ and $t$ refers to the steps in the symmetry breaking scheme. (In the $(d-1)$ transverse directions, periodic boundary conditions have been imposed, and the mean field solution is independent of

those transverse space variables.) In terms of those functions the action reads

$$
S/n = \frac{L^{d-1}}{4} \int_0^L dz \left[ \int_0^1 dt \left\{ -\frac{\partial Q}{\partial z} \frac{\partial}{\partial z}(\dot{p}Q) \right.\right.
$$

$$
\left. + \dot{p} \left( \frac{\tau}{2}Q^2 + \frac{u}{12}Q^4 \right) \right\} - \frac{w}{6} \int_0^1 dt \dot{p}(t,z) \left( p(t,z)Q^3(t,z) \right.
$$

$$
\left.\left. + 3Q^2(t,z) \int_t^1 ds \dot{p}(s,z)Q(s,z) \right) \right] \quad (9)
$$

$(\dot{p}, \dot{Q})$ denote derivatives with respect to $t$.

In the bulk, with non twisted boundary conditions, this action is manifestly a function of $Q(p)$ alone, and not separately of $Q(t)$ and $p(t)$. Indeed the extrema of this free energy are given as solutions of

$$
A(t) = \tau Q + \frac{u}{3}Q^3 - \frac{w}{2} \left( pQ^2 + \int_0^t ds \frac{dp}{ds}Q^2(s) \right.
$$

$$
\left. + 2Q \int_t^1 ds \frac{dp}{ds}Q(s) \right) = 0. \quad (10)
$$

and Parisi's solution is

$$
Q(t) = \frac{w}{2u}p(t) \quad \text{for} \quad 0 < t < x_1
$$

$$
Q(t) = Q_1 \quad \text{for} \quad x_1 < t < 1 \quad (11)
$$

with the Edwards-Anderson order parameter $Q_1$ defined by

$$
\tau + uQ_1^2 - wQ_1 = 0. \quad (12)
$$

With the twisted boundary conditions we obtain two variational equations for $Q(t,z)$ and $p(t,z)$, which read

$$
\dot{Q}A + \frac{1}{4}\frac{\partial}{\partial t}\left( Q\frac{\partial^2 Q}{\partial z^2} \right) = 0
$$

$$
\dot{p}A + \frac{1}{4}\frac{\partial^2}{\partial z^2}(\dot{p}Q) + \frac{1}{4}\dot{p}\frac{\partial^2 Q}{\partial z^2} = 0 \quad (13)
$$

in which $A$ is defined in (10).

To lowest order in the imposed twist $h(t)$ one checks easily that the solution is

$$
p(t,z) = \frac{2u}{w}Q(t,z) = t + \frac{z}{L}h(t) \quad (14)
$$

for $0 < t < \tilde{x}$ ($\tilde{x}$ is the end of the support of $h(t)$ and then the solution is the bulk one for $t > \tilde{x}$. To second order in $h(t)$ though, the bulk proportionality of $p$ and $Q$ is lost).

The incremental free energy, which follows from this twisted solution (compared to the bulk one), comes purely from the kinetic energy (since the bulk relation $p(t,z) =$

$\frac{2u}{w}Q(t,z)$ still holds). The final mean field result is, at lowest order in $h(t)$,

$$
\Delta F_{twist} = -\lim_{n\to 0} \frac{1}{n}(Z^n_{\text{twisted}} - Z^n_{\text{bulk}})
$$

$$
= -\left( \frac{w}{2u} \right)^2 L^{d-2} \int_0^{t_1} dt h(t)(h(t) + t\dot{h}(t))
$$

$$
= -\frac{1}{2}\left( \frac{w}{2u} \right)^2 L^{d-2} \int_0^{t_1} dt\ h^2(t). \quad (15)
$$

This result holds above the upper critical dimension $d_u = 6$ but, contrary to usual ordered-disorder transitions, we shall see that the $L$-dependence is modified by fluctuations. We now proceed to the one-loop computation.

## 3 One-loop fluctuations around mean field

The theory is now extended in dimension $d = 6 - \epsilon$, dimensionally regularized, and later renormalized.

1. Consider first the *Replicon* sector where the fluctuation matrix may be fully diagonalized (for a review of fluctuations beyond mean field see [8]). Its continuation to the bulk would write, in the continuum limit,

$$
F^{(R)}_{loop} = \frac{L^d}{2} \left[ -\int_0^1 \frac{dt}{2} \int_t^1 \frac{dk}{k} \frac{\partial}{\partial k} \int_t^1 \frac{dl}{l} \frac{\partial}{\partial l} \right]
$$

$$
\times \int \frac{d^d p}{(2\pi)^d} \log \left( p^2 + \frac{g}{2}(k^2 + l^2 - 2t^2) \right) \quad (16)
$$

where we have used the notation

$$
g = \frac{w^2}{2u}. \quad (17)
$$

In (16) the first bracket comes from the multiplicity of the replicon modes with associated eigenvalues $p^2 + \Delta_0(k,l;t)$ as in the argument of the logarithm

$$
\Delta_0(k,l;t) = \frac{g}{2}(k^2 + l^2 - 2t^2). \quad (18)
$$

Under the twist (14), the above bulk result (16) is changed in two ways. First the argument of the logarithm is to be replaced by

$$
\log \left( q_T^2 - \frac{\partial^2}{\partial z^2} + \Delta_0 + \frac{z}{L}\Delta_1 \right) \quad (19)
$$

with

$$
\Delta_1 = g[kh(k) + lh(l) - 2th(t)]. \quad (20)
$$

In fact there is also a quadratic term $(z/L)^2\Delta_2$, that has been omitted here since at one-loop it cancels through the bulk subtraction.

Expanding now the logarithm to second order in $\Delta_1$ (in order to collect the quadratic terms in $h(t)$), we obtain as twist contribution the term

$$\frac{L^{d-1}}{4} \int_0^1 \frac{dt}{2} \int_t^1 \frac{dk}{k} \frac{\partial}{\partial k} \int_t^1 \frac{dl}{l} \frac{\partial}{\partial l} \int \frac{d^{d-1}q_T}{(2\pi)^{d-1}}$$

$$\int_0^L dz \frac{z}{L} \int_0^L dz' \frac{z'}{L} \left( \frac{1}{\pi} \int_{-\infty}^{+\infty} dK \frac{\sin Kz \sin Kz'}{q_T^2 + K^2 + \Delta_0} \right)^2 \Delta_1^2. \tag{21}$$

In this expression boundary conditions at the two end planes $z = 0$ and $z = L$ have been taken into account; indeed the fluctuating part of the field vanishes at those boundaries, leading to the appropriate basis $\sin \pi \frac{mz}{L}$ with $m = 1, 2, \cdots$

Secondly, the twist (14) changes also the multiplicity itself. This is taken into account *via* an identity expressing the reparametrization invariance under a *z-independent* shift $\Delta_0 \to \Delta_0 + \Delta_1$. As a result the total twist contribution is then obtained by replacing in (21) $\frac{zz'}{L^2}$ by $\frac{zz' - 1/2(z^2 + z'^2)}{L^2}$.

Performing the K-integration on the modified (21) one obtains

$$\Delta F_{twist}^R = -\frac{L^{d-3}}{8} \int_0^1 \frac{dt}{2} \int_t^1 \frac{dk}{k} \frac{\partial}{\partial k} \int_t^1 \frac{dl}{l} \frac{\partial}{\partial l}$$

$$\times \int_0^L dz \int_0^L dz' \left[ \frac{e^{-M|z-z'|} - e^{-M(z+z')}}{2M} \right]^2 \Delta_1^2 \tag{22}$$

in which

$$M = q_T^2 + \Delta_0. \tag{23}$$

Performing the $z, z'$ and finally $q_T$ integrations, to gather poles in $\varepsilon$ and logarithms, one obtains:

$$\Delta F_{twist}^{(R)} =$$

$$-\frac{1}{12} L^{d-2} S_d g^2 \int_0^{\tilde{x}} dt \, h^2(t) \left[ \frac{1}{\varepsilon} + \log L + \cdots \right], \tag{24}$$

with

$$S_d = \frac{2}{(4\pi)^{d/2} \Gamma(d/2)}. \tag{25}$$

It is a strong argument in favour of the consistency of the calculation to see that the one-loop contribution is, like mean-field, proportional to the integral $\int dt \, h^2(t)$. Indeed, otherwise the fluctuations would not be renormalized by a simple change in the coupling constant. In the intermediate steps this final form is far form obvious. In particular it involves the unexpected identity

$$\int_0^1 \frac{dt}{2} \int_t^1 \frac{dk}{k} \frac{\partial}{\partial k} \int_t^1 \frac{dl}{l} \frac{\partial}{\partial l} \Delta_1^2(k, l; t) = g^2 \int_0^{\tilde{x}} dt \, h^2(t). \tag{26}$$

2. In the *Longitudinal-Anomalous* (L-A) sector, the fluctuation matrix can only be diagonalized by blocks [8] (with blocks of size $(R+1) \times (R+1)$, $R$ being equal to the number of steps of replica symmetry breaking). In the Parisi limit, in which $R$ goes to infinity, the L-A contribution to the bulk free energy writes [8]

$$F_{loop}^{(LA)} = \frac{L^d}{2} \left[ \int_0^1 \frac{dk}{k} \frac{\partial}{\partial k} \right] \int \frac{d^d \boldsymbol{p}}{(2\pi)^d} \mathrm{tr}$$

$$\times \log \left[ 1 + \frac{1}{p^2 + \Delta_0(k, t; t)} B_k(t, s) \right] \tag{27}$$

where, again, the first bracket comes from the new multiplicity of the L-A modes. Besides we have the $(t, s)$ matrix

$$B_k(t, s) = g(\mathrm{Inf}(t, s))[\Theta(k - s) + k\delta(k - s) + 2\Theta(s - k)]ds \tag{28}$$

and, as in (18)

$$\Delta_0 = g(k^2 - t^2)/2 \qquad t < k$$

$$= 0 \qquad t \geq k. \tag{29}$$

The calculation is more involved here, but we proceed as in the replicon sector, collecting quadratic terms in $h$ when performing the twist transform as in (14) or in $\Delta_0 \to \Delta_0 + (z/L)\Delta_1$. As above we keep only terms *quadratic* in the propagator $(q_T^2 - d^2/dz^2 + \Delta_0)^{-1}$, higher order terms being, at one loop, ultra-violet convergent. In contrast to the replicon sector (where only the twist of $\Delta_0$ into $\Delta_0 + (z/L)\Delta_1$ contributed) we need here to take care of the twists over the matrix elements $B_k(t, s)$ and over the multiplicity. Altogether, complicated expressions rearrange themselves to give

$$\Delta F_{twist}^{(LA)} = \frac{1}{6} L^{d-2} S_d g^2 \int_0^{\tilde{x}} dt \, h^2(t) \left[ \frac{1}{\varepsilon} + \log L + \cdots \right]. \tag{30}$$

Notice that, at one-loop, there is no contribution in $\log(\tau)$ *i.e.* involving the mass $wQ_1 \simeq \tau$ (as in (12)). The reason is that neither $Q_1$ (*i.e.* $x_1$), nor $p_0 = n$, fluctuate under reparametrization. Even as a correction term to the contribution in $(zh/L), (z'h/L)$, it would take to expand the logarithm to third order before $Q_1$ showing up.

# 4 Renormalization and scaling

Before proceeding, one has to take into account the fact that the $\frac{u}{12} \sum_{ab} \phi_{ab}^4$ coupling is irrelevant, and "dangerous" since the fluctuations make it singular below dimension eight [8,9]. Indeed in a pure $\frac{w}{6} \mathrm{tr} \phi^3$ theory, the one-loop contribution has the effect of replacing $u$ by

$$u \to u + 12w^4 \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^4(p^2 + 2\tau)^2} =$$

$$u + 6S_d w^4 (2\tau)^{-(1+\varepsilon/2)}. \tag{31}$$

We thus have for the twist free energy

$$\Delta F_{twist} = -\frac{S_d}{288w^6}L^{d-2}\tau^{2+\varepsilon}$$
$$\times \int_0^{\tilde{x}} dt\, h^2(t)\left[1 - \frac{2}{3}w^2\left(\frac{1}{\varepsilon} + \log L\right) + \cdots\right] \quad (32)$$

(a factor $S_d$ has been included in $w^2$). It is now crucial to verify that the replacement of the coupling constant and temperature by their renormalized counterpart $w_R, \tau_R$ rids us of the $1/\varepsilon$ poles. The computation of those renormalizations is easily done in the paramagnetic phase. It gives [11]

$$\tau_R = \tau\left[1 - \frac{4w^2}{\varepsilon} + O(w^4)\right]Z$$
$$w_R^2 = w^2\left[1 - \frac{4w^2}{\varepsilon} + O(w^4)\right]Z^{3/2}$$
$$Z = \left[1 + \frac{2w^2}{3\varepsilon} + O(w^4)\right] \quad (33)$$

from which follows

$$\frac{\tau^2}{w^6} = \frac{\tau_R^2}{w_R^6}\left[1 + \frac{2w_R^2}{3\varepsilon} + O(w_R^4)\right]. \quad (34)$$

The $1/\varepsilon$ pole is thus exactly cancelled and we end up with

$$\Delta F_{twist} = -\frac{S_d}{288w_R^6}L^{d-2}\tau_R^{2+\varepsilon}$$
$$\times \int_0^{\tilde{x}} dt\, h^2(t)\left[1 - \frac{2}{3}w_R^2 \log L + \cdots\right]. \quad (35)$$

If we substitute to $w_R$ the fixed point $w^*$, zero of the $\beta$-function

$$\beta(w_R) = -\frac{\varepsilon}{2}w_R + w_R^3 + O(w_R^5) \quad (36)$$

one obtains that, to this order, the result exponentiates to

$$\Delta F_{twist} = -\frac{S_d}{288w_R^6}\tau_R^{2+\varepsilon}L^{d-2-\varepsilon/3}\int_0^{\tilde{x}} dt\, h^2(t). \quad (37)$$

Introducing the usual critical exponents $\eta$ and $\nu$, whose $\varepsilon$-expansions are known to be [11]

$$\eta = -\frac{1}{3}\varepsilon + O(\varepsilon^2) \qquad \nu = \frac{1}{2}\left(1 + \frac{5}{6}\varepsilon + O(\varepsilon^2)\right), \quad (38)$$

one may write to this order

$$\Delta F_{twist} \sim -\tau_R^{\nu(d-2+\eta)}L^{d-2+\eta}. \quad (39)$$

This is reasonable since it gives the final twist free energy as a function of $L/\xi$ to a power, which is the expected scaling form for the ordinary order-disorder transitions:

$$\Delta F_{twist} \sim -\left(\frac{L}{\xi}\right)^{d-2+\eta}. \quad (40)$$

There are two differences though with ordinary transitions.

- First the power of $L/\xi$ is non-canonical. We thus verify, to the order of one-loop, an extended form of scaling, appropriate when the soft transverse modes are not isolated, but being at the bottom of a gapless band, they are no longer infra-red free. Consequently the fluctuations exhibit a logarithmic dependence as the size and the propagator develops a (*negative*) anomaly $\eta$. At the lower critical dimension $d_c$, the twist free energy should vanish and thus $d_c$ should be the solution of the equation

$$d_c = 2 - \eta(d_c). \quad (41)$$

  This same answer had been anticipated earlier on the basis of scaling arguments applied to the null overlap replicon sector [10]. Using numerical estimates for $\eta$ [13], one gets from (41) a value of $d_c$ close to 2.5, whereas a self consistent mean-field approach for the twisted free-energy of two copies, surprisingly yields [14] exactly 5/2.

- The sign of this twisted free energy is negative. This will be discussed below.

## 5 Discussion

The calculation of fluctuations around the background of a twisted mean field solution is renormalizable, as expected, in spite of its spatial non-uniformity, as already checked in the simple O(N)-model [6]. However contrary to the simple ferromagnetic transitions, the influence of this twist on the size dependence manifests itself by a logarithmic dependence in $L$ at one-loop which exponentiates to a negative anomaly. This generates an increase of the lower critical dimension.

However another difference with ferromagnetic transitions is the sign dependence of the twist on the free energy: the twisting leads here to a decrease in the free energy. This is not in contradiction with the principles of thermodynamics. A similar situation could occur with an antiferromagnet, since there as well, the free energy may be lowered by imposing external fields of opposite signs at two ends of the sample. In such a circumstance, one expects that the system would spontaneously breaks spatial uniformity and develop a space dependence in the "gauge" choice of the replica symmetry breaking. Unfortunately we are not aware of any conjugate field to the gauge choice $p(t, z)$ which could be imposed to improve the mean field starting point.

However no conclusion can really be drawn at this stage from this negative sign. For a long time it had been suspected that the free energy should be maximised in the zero-replica limit. In the case of the infinite range Sherrington-Kirkpatrick model this has now been rigorously established by Guerra [16]. In our calculations, in spite of the positivity of the Hessian eigenvalues, characterizing fluctuations around mean field solution, their multiplicities become negative in the zero-replica limit. Therefore the free energy may be lowered by some excitations, without contradicting the thermodynamic stability

of the mean field solution. The only unambiguous result remains the appearance of logarithmic fluctuations involving the size $L$ of the system, which push the lower critical dimension above two.

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